Nonlocally-induced (fractional) bound states: Shape analysis in the infinite Cauchy well

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(Dated: August 26, 2016)

Fractional (Lévy-type) operators are known to be spatially nonlocal. This becomes an issue if confronted with a priori imposed exterior Dirichlet boundary data. We address spectral properties of the prototype example of the Cauchy operator $(-\Delta)^{1/2}$ in the interval $D = (-1, 1) \subset R$, with a focus on functional shapes of lowest eigenfunctions and their fall-off at the boundaries of D. New high accuracy formulas are deduced for approximate eigenfunctions. We analyze how their shape reproduction fidelity is correlated with the evaluation finesse of the corresponding eigenvalues.

I. FRACTIONAL LAPLACIANS: R VERSUS $D \subset R$.

The Fourier integral $\frac{1}{\sqrt{2\pi}} \int_R |k|^\mu \tilde{f}(k) e^{-\imath kx} dk = -\partial_\mu f(x)/\partial |x|^\mu = |\Delta|^{\mu/2} f(x)$ is commonly interpreted as a definition of a fractional derivative of the μ-th order for $\mu \in (0, 2)$. The notation $-(-\Delta)^{\mu/2} = -|\Delta|^{\mu/2}$ refers to a fractional Laplacian of order $\mu/2$ (restricted to dimension one i.e. to R) and two versions of a fractional dynamics (dimensional constants being scaled away): semigroup $\exp(-t|\Delta|^{\mu/2}) f$ and unitary $\exp(-it|\Delta|^{\mu/2}) f$, [1]-[5]. Here $\tilde{f}$ stands for a Fourier transform of $f \in L^2(R)$ and $g(k) = |k|^\mu \tilde{f}(k)$ is presumed to be $L^2(R)$-integrable.

Apart from the unperturbed (free) case, the Fourier (multiplier) representation of the fractional dynamics has proved useful if an infinite or periodic support is admitted for functions in the domain, [2]. For the simplest quadratic ($\sim x^2$) perturbation of the fractional Laplacian (the fractional oscillator problem), a complete analytic solution has been found in the specialized Cauchy oscillator case [6, 7], by resorting to Fourier space methods.

For more complicated perturbations, and likewise for a deceivingly simple problem of the fractional Laplacian in a bounded (spatial) domain, standard Fourier techniques seem to be of a doubtful or limited use, [2]. A fully-fledged spatially nonlocal definition of the fractional Laplacian appears to be better suited to handle such problems, [8]-[15]. See e.g. also [16] for a construction of Cauchy semigroups which arise from various perturbations of the Cauchy operator by bounded or locally bounded positive functions (i.e. external potentials).

A. $|\Delta|^{\mu/2}$ on R.

The fractional Laplacian $-|\Delta|^{\mu/2}$, $\mu \in (0, 2)$ is a pseudo-differential (integral) operator and its action on a function from the $L^2(R)$ domain is defined as follows:

$$-|\Delta|^{\mu/2} f(x) = \int_R [f(x+y) - f(x) - \frac{y \nabla f(x)}{1+y^2}] \, \nu_\mu(dy), \tag{1}$$

where $\nu_\mu(dx)$ stands for the Lévy measure. This definition is commonly simplified by employing the Cauchy principal value of the involved integral (evaluated relative to the singular points of integrands)

$$|\Delta|^{\mu/2} f(x) = -\int_R [f(x+y) - f(x)] \nu_\mu(dy) = -\frac{\Gamma(\mu+1)\sin(\pi\mu/2)}{\pi} \int_R \frac{f(z) - f(x)}{|z-x|^{1+\mu}} dz \tag{2}$$

Here, the Lévy measure $d\nu_\mu$ has been made explicit and we point out a change of the integration variable $y \to z + x$. The Fourier representation of the the integral formula (2) takes the form

$$|\Delta|^{\mu/2} f(x) = -\frac{\Gamma(1+\mu)\sin\frac{\pi\mu}{2}}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{-\imath kx} dk \int_{-\infty}^{\infty} \frac{(e^{-\imath ky} - 1)dy}{|y|^{1+\mu}}. \tag{3}$$

The integral over dy, presuming its existence (which is not the case for $\mu = 1$) can be directly evaluated

$$\int_{-\infty}^{\infty} \frac{(e^{-\imath ky} - 1)dy}{|y|^{1+\mu}} = 2|k|^\mu \Gamma(-\mu) \cos\frac{\pi\mu}{2}. \tag{4}$$

Since Eq. (3) is undoubtedly valid for all $\mu \in (0, 1) \cup (1, 2)$, we can substitute back an outcome of (4) and employ an identity $\Gamma(1+\mu)\Gamma(-\mu) = -\pi/\sin(\pi\mu)$ (remember that the function $\Gamma(-\mu)$ has simple poles at $0, -1, -2$). Accordingly,

two potentially divergent entries compensate each other and the limit $\mu \to 1$ is now legitimate. Thus $|\Delta|^{\mu/2} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |k|^\mu \tilde{f}(k) e^{-\imath kx} dk$ for all $\mu \in (0,2)$, and $|k|^\mu$ is a Fourier multiplier of $|\Delta|^{\mu/2}$ on R, as anticipated, see [5] and [17].

The fractional Laplacian $|\Delta|^{\mu/2}$ extends to a self-adjoint operator in $L^2(R)$ and induces a strongly continuous contraction semigroup $\exp(-t|\Delta|^{\mu/2})$ whose Fourier multiplier equals $\exp(-t|k|^\mu)$.

B. Interlude: $-\Delta$ in $D \subset R$.

The Hamiltonian-type expression $H = -\Delta + V$, with $V(x) = 0$ for $x \in D = (-1,1) \subset R$, is an encoding of the Laplacian with the the Dirichlet boundary conditions (so-called zero exterior condition on $R \setminus D$) imposed on $L^2(R)$ functions $f(x)$ in the domain of H: $f(x) = 0$ for $|x| \geq 1$. The problem is that so defined operator H, if restricted to a domain containing solely functions $f \in L^2(R)$ with a support in D, is not a self-adjoint operator in $L^2(R)$ (to this end we need to admit $C_0^\infty(R)$ as a proper domain).

We recall that $C_0^\infty(R)$ comprises infinitely differentiable functions that are compactly supported in R. The notation $C_0^\infty(D)$ refers to a definite choice of the support to be $D \subset R$. The differential operator $-\Delta$ when acting in $C_0^\infty(D)$ (we keep $D = (-1,1)$) defines a symmetric operator in $L^2(D)$. The problem of self-adjoint extensions in this case is a classic, c.f. [18]. One of them, with a domain $D(H) = \{AC^2[-1,1], f(-1) = 0 = f(1)\}$ corresponds to a standard (quantum mechanical) infinite well problem; the AC^2 notation refers to an absolute continuity of f which gives meaning to the second derivative of f (of relevance at the boundary points of D).

The spectral solution gives rise to the $L^2(D)$ orthonormal eigenbasis, composed of real functions $f_n(x)$, $n = 1,2,...$ such that $f(x) = 0$ for $|x| \geq 1$. More explicitly: $f_n(x) = cos(n\pi x/2)$ for n odd and $sin(n\pi x/2)$ for n even, while respective eigenvalues read $E_n = (n\pi/2)^2$. It is clear that any $f \in L^2(D)$, in the domain of the infinite well Hamiltonian, may be represented as $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$, e.g. in the form of (trigonometric) Fourier series.

At this point it is useful to mention that, quite independently of the self-adjointness issue, in $L^2(D)$ we have two inequivalent ways of making the Fourier analysis. If $R \backslash D$ is neglected and $L^2(D)$ is considered on its own (without any reference to $L^2(R)$), then we can employ the previously mentioned Fourier series (e.g. the infinite well trigonometric eigenbasis).

We shall pay more attention to the alternative approach. Namely, if $L^2(D)$ is considered as a subspace of $L^2(R)$, then for any $0 \neq f \in L^2(D)$ we know that $\tilde{f} \in L^2(R)$ but $\tilde{f}$ no longer belongs to $L^2(D)$.

In fact, for any $f \in C_0^\infty(R)$ its Fourier transform $\tilde{f}$ is an entire function (analytic on the whole complex plane). It does not vanish anywhere in R, except at infinity. If $\tilde{f}$ would vanish on $R \setminus D$, then necessarily it would be vanishing on D as well.

Since f is infinitely differentiable it follows that $\tilde{f}$ goes to zero along the real axis faster than the inverse of any polynomial. These properties hold true irrespective of the choice of the compact supporting interval $D \subset R$. Accordingly, $\mathcal{F}L^2(D) \subset L^2(R)$, but $\mathcal{F}L^2(D) \bigcap L^2(D) = \emptyset$.

Under the infinite well conditions, $|k|^2$ still retains some of the Fourier multiplier (of $-\Delta$) features. This view is supported by an approximation of the infinite well problem by a sequence of deepening finite wells, [18]. The convergence can be quantified in the $L^2(R)$ norm.

Useful examples worked out in [19, 20], at the first glance, indicate an undoubtful relevance of k and k^2 multipliers. However, an the emergence of technical problems becomes conspicuous, if the multiplier property is to be elevated to eigenfunctions which are not $C_0^\infty(D)$.

In fact, for the infinite well eigenfunctions $f_n(x)$ ($L^2(D)$-normalized), their $L^2(R)$ Fourier transforms $\tilde{f}_n(k) = (2\pi)^{-1/2} \int_D f_n(x) \exp(-ikx)\, dx$ can be directly evaluated. We have $(f,f)_{L^2(D)} = (\tilde{f}, \tilde{f})_{L^2(R)}$ and there holds $\int_R |\tilde{f}_n(k)|^2\, k^2\, dk = (n\pi/2)^2 = (f_n, (-\Delta f_n))$. Likewise $(f_n, (-i\nabla f_n)) = \int_R |\tilde{f}_n(k)|^2\, k\, dk = 0$ for all $n \geq 1$.

One should be aware that the existence of mean values of Fourier multipliers does not mean that we can execute an inverse Fourier transform of e.g. the function $g(k) = |k|^2 \tilde{f}$ and retrieve $-\Delta f(x)$ as an image-function in $L^2(D)$. Actually, the inverse Fourier transform does not exist in this case, unless we adopt a weaker definition [17]. We note that $-\Delta_D f(x)$, at the boundaries ± 1 of D, needs to be interpreted as a generalized function (distribution), and has a meaning only if smoothed out by a suitable test function. Incidentally, the eigenfunctions f_n appear to play such a smoothing role, while evaluating mean values.

C. Cauchy operator in $D \subset R$: from $L^2(R)$ to $L^2(D)$.

The previously indicated jeopardies related to the Fourier multiplier definition (in case of $\mu = 2$) surely extend to fractional Laplacians. Therefore, in the presence of spatial restrictions upon domains of nonlocal operators, we choose

to investigate their properties directly in configuration space with no recourse to Fourier transforms (and multipliers).

The action of the Cauchy operator on $C_0^\infty(R)$ functions (differentiable, with continuous derivatives of all orders and compactly supported) is given by a specialized version of Eq. (2):

$$|\Delta|^{1/2} f(x) = \frac{1}{\pi}\int_R \frac{f(x)-f(x+z)}{z^2}dz = \frac{1}{\pi}\int_R \frac{f(x)-f(z)}{|z-x|^2}dz, \quad x\in R, \tag{5}$$

and clearly has a Fourier representation with the multiplier $|k|\,\tilde{f}(k)$. We note in passing that so defined $(-\Delta)^{1/2}$ extends to an unbounded self-adjoint operator in $L^2(R)$.

The Cauchy operator $|\Delta|^{1/2}$ if restricted to a domain comprising solely $L^2(R)$ functions with a support in D and vanishing on $R\backslash D$ is not a self-adjoint operator in $L^2(R)$. However, if we consider the action of $|\Delta|^{1/2}$ on test functions $f\in C_0^\infty(D)$, then the restriction $|\Delta|_D^{1/2} f$ of $|\Delta|^{1/2}$ to D is interpreted as the Cauchy operator with the zero (Dirichlet) exterior condition on $R\backslash D$ and is known to extend to a self-adjoint operator in $L^2(D)$, [8].

The passage from $C_0^\infty(R)$ to $C_0^\infty(D)$ ultimately amounts to disregarding any $R\backslash D$ contribution implicit in the formal definition (5) and makes the usage the Fourier multiplier representation either clumsy or redundant.

Let us consider the D versus $R\backslash D$ interplay in more detail, by considering the action of $|\Delta|^{1/2}$ on these $C_0^\infty(R)$ functions which are actually supported in D. The major problem we wish to address is an explicit spatial form of the eigenvalue problem for $|\Delta|_D^{1/2}$, interpreted as $|\Delta|_D^{1/2} f = E\, f$ where $E\in R^+$ is an eigenvalue and $f\in L^2(D)$. No closed analytic solutions are here available and various approximate methods were invented to optimize approximations of "true" eigenvalues and shapes of respective "true" eigenfunctions, [8]-[15].

Each known to date approximate eigenfunction $\psi(x)$, [8, 11, 15], by construction belongs to the domain of $|\Delta|_D^{1/2}$ and obeys exterior Dirichlet boundary data. However, generically $|\Delta|_D^{1/2}\psi \in L^2(D)$ no longer respects those data, while such a property is definitely expected from an acceptable $L^2(D)$ approximation of the right-hand -side of the eigenvalue formula $|\Delta|_D^{1/2} f = E\, f$.

This problem is typically bypassed in the mathematical literature, where one considers the spectral problem of finding an eigenfunction (or its optimal approximation) in a weaker form. This (somewhat relaxed) approach to the spectral problem stems from the adopted definition of the action of $|\Delta|_D$ in its $C_0^\infty(D)$ domain. Namely, one demands that for f in the $L^2(D)$ domain of $|\Delta|_D$ there exists $|\Delta|_D\, f\in L^2(D)$ such that $(g, |\Delta|_D\, f)_{L^2(D)} = (|\Delta|\, g, f)_{L^2(D)}$ holds true for $g\in C_0^\infty(D)$, [21, 22].

Let us tentatively consider the action of $|\Delta|^{1/2}$ on $C_0^\infty(R)$ functions $\psi(x)$, supported in $D=(-1,1)$. Accordingly, for all $x\in(-1,1)$ we have:

$$A_D\psi(x) = \frac{2}{\pi}\frac{\psi(x)}{1-x^2} + \frac{1}{\pi}\int_{-1-x}^{1-x} \frac{\psi(x)-\psi(x+y)}{y^2}dy. \tag{6}$$

The integral in (6) should be understood as the Cauchy principal value evaluated with respect to 0, according to $\int_{-1-x}^{1-x} = \lim_{\varepsilon\to 0}\left[\int_{-1-x}^{-\varepsilon} + \int_{-\varepsilon}^{1-x}\right]$.

Let us change the integration variable $y=t-x$ in Eq. (6). We have:

$$A_D\psi(x) = \frac{2}{\pi}\frac{\psi(x)}{1-x^2} + \frac{1}{\pi}\int_{-1}^{1} \frac{\psi(x)-\psi(t)}{(t-x)^2}dt \tag{7}$$

where the $R\setminus D$ and D contributions are now clearly isolated. The Cauchy principal value of the integral in Eq. (7) is no longer evaluated with respect to 0, but with respect to x. The integral expression in Eq. (7) which is now restricted to $t\in(-1,1)$ while $x\in(-1,1)$ by definition, is a special (Cauchy) case of so-called regional fractional Laplacian for D, [12–14].

II. TRIAL ANALYTIC FORMS OF APPROXIMATE EIGENFUNCTIONS.

A. The ground state function.

In the present paper, we shall employ Eq. (6) as the definition of the Cauchy operator in action on functions $\psi\in\mathcal{D}(D)$, in the spatially restricted domain $D=(-1,1)$. We no longer require ψ to belong to $C_0^\infty(D)$, we need however $\psi(x)$ to be infinitely differentiable in D and identically vanish for $|x|\geq 1$. Let us address the eigenvalue problem $A_D\psi(x) = E\psi(x)$, whose approximate solution will be sought for subsequently.

The term $\frac{2}{\pi}\frac{1}{1-x^2}$ in Eq. (6) is a sum of the geometric series $\frac{2}{\pi}(1+x^2+x^4+\ldots)$, $x \in (-1,1)$. Therefore it seems natural to assume that a solution $\psi(x)$ of $A_D\psi(x) = E\psi(x)$ might be represented in the form of the power series $\psi(x) = \sum_{n=0}^{\infty} c_n x^n$ as well, with a proviso that $\psi(x)$ needs to be identically 0 at the boundaries ± 1 of D. Thus we allow $\psi(x)$ to have a domain $\bar{D} = [-1,1]$.

The ground state is an even concave function, [9, 10], therefore we actually expect that $\psi(x) = \sum_{n=0}^{\infty} c_{2n}x^{2n}$. Since $\psi(\pm 1) = 0$, there holds $c_0 + c_2 + \ldots = 0$. Inserting ψ to the eigenvalue formula and keeping $x \in (-1,1)$ we get

$$\frac{1}{\pi}\int_{-x-1}^{-x+1} \frac{-\psi'(x)z - \psi''(x)\frac{z^2}{2!} - \psi'''(x)\frac{z^3}{3!} - \ldots}{z^2}dz + \frac{2}{\pi}\frac{1}{1-x^2}\psi(x) = E\psi(x). \tag{8}$$

presuming that the integral (Cauchy principal value) and series summation can be interchanged, next setting $x = 0$, we arrive at the series expansion which defines the ground state eigenvalue E, given $\psi(x)$:

$$-\frac{2}{\pi}\left(\frac{c_2}{1} + \frac{c_4}{3} + \frac{c_6}{5} + \ldots\right) + \frac{2}{\pi}c_0 = Ec_0. \tag{9}$$

With $c_0 \neq 0$, we have

$$E = \frac{2}{\pi}\left[1 - \frac{1}{c_0}\left(\frac{c_2}{1} + \frac{c_4}{3} + \frac{c_6}{5} + \ldots\right)\right]. \tag{10}$$

The series converge, which follows (via the D'Alembert criterion) from the convergence of $\sum_{n=0}^{\infty} c_{2n}x^{2n} = \psi(x)$ for all $x \in [-1,1]$.

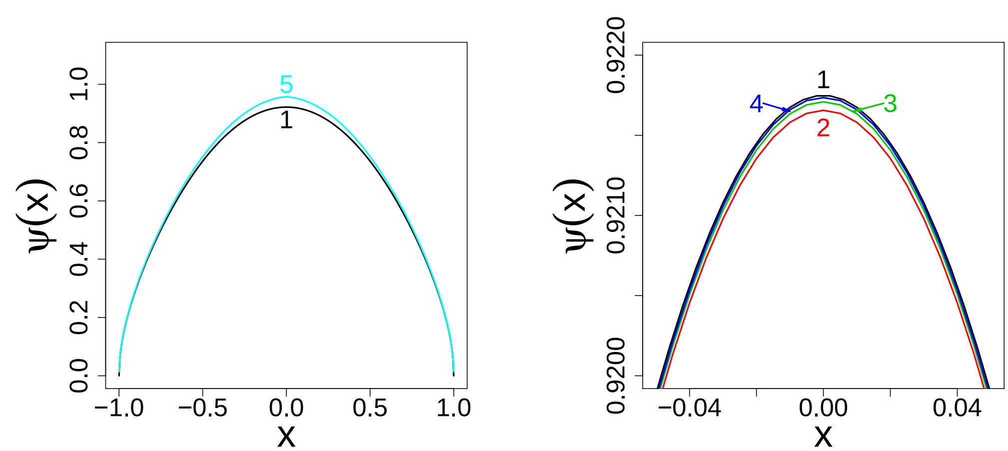

FIG. 1: Approximate ground states for Cauchy wells. Numbers refer to: 1- infinite well $\psi(x)$ of Eq.(11); 2, 3, 4 - finite wells with depths $V_0 = 5000, 10000, 20000$ respectively, [15]; 5 - infinite well proposal of [8]. In the right panel, the curve 5 is out of the frame.

By independent arguments, we know that the ground state function should be close (loosely speaking) to the $\cos(x)$ while away from the boundaries of D and $(1-x^2)^{1/2}$ in the close vicinity of the boundaries, [2, 8, 15]. Let us consider the trial approximation of ground state function, given in an analytic form:

$$\psi(x) = C\sqrt{(1-x^2)\cos(\alpha x)}, \tag{11}$$

where the coefficient α has been adjusted to differ slightly from $\frac{\pi}{2} - \frac{\pi}{8}$, known to be the leading term in the asymptotic eigenvalue formula for the infinite Cauchy well, [8]:

$$\alpha = \frac{1443}{4096}\pi = \left(\frac{\pi}{2} - \frac{\pi}{8}\right) - \frac{\pi}{64} - \frac{\pi}{256} - \frac{\pi}{512} - \frac{\pi}{1024} - \frac{\pi}{4096}, \tag{12}$$

$C = 0.921749$ being the $L^2(D)$ normalization constant.

In the recent paper [15] we have introduced an algorithm for evaluating approximate eigenfunctions of *finite* Cauchy wells of arbitrary depth. The idea was to implement as close approximation of the infinite well spectral properties in terms of those for very deep finite wells. In Fig. 1 we have depicted approximate ground state functions for finite wells of depths $V_0 = 5000, 10000, 20000$, an approximate ground state for the infinite Cauchy well proposed in Ref. [8] and our approximate formula $\psi(x)$ of Eq. (11) for the infinite well.

In the left panel, curves for $V_0 = 5000, 10000, 20000$ and that for $\psi(x)$ are graphically indistinguishable in the adopted scales, while the proposal of [8] is conspicuously different. In the right panel a vicinity of the maximum has been enlarged and the proposal of [8], in the adopted scale, is out of frame.

Let us expand the approximate ground state ψ of Eq. (11) into power series $\psi(x) = \sum_{n=0}^{\infty} c_{2n} x^{2n}$ with $x \in [-1, 1]$. The expansion coefficients can be explicitly identified and we reproduce numerical values for first few of them:

$$\begin{array}{lllll} c_0 = 0.921749, & c_2 = -0.743145, & c_4 = 0.011510, & c_6 = -0.020710, & c_8 = -0.015567, \\ c_{10} = -0.012318, & c_{12} = 0.009969, & c_{14} = -0.008234, & c_{16} = -0.006922, & c_{18} = -0.005910. \end{array} \tag{13}$$

Although $\psi(x)$ is not a "true" eigenfunction but an approximation of the ground state, by employing merely first 10 expansion coefficients in the series expansion Eq. (10), we obtain the very rough outcome $E = 1.15318$. The ground state eigenvalues have been approximated by other (more accurate) methods and, up to four decimal digits we have e.g.: 1.1577 according to [8], 1.1573 according to [15].

We point out that a convergence of the series (10) is very slow. To get more accurate approximation of the eigenvalue associated with the approximate ground state $\psi(x)$ (11) we need to account for much larger number of expansion coefficients c_{2n}.

B. Analysis of $A_D\psi(x)$.

At the moment we are not that much interested in producing high accuracy approximate formulas (this issue will be addressed subsequently in the present paper). The analytic expression Eq. (11) for $\psi(x)$ is extremely useful for another purpose.

Namely, we can make explicit the action od A_D upon functions with definite geometric shapes and analyze not only how much $A_D\psi(x)$ deviates from $\psi(x)$ and ultimately from $E\,\psi(x)$ (with a properly adjusted value E), but also the boundary behavior of those functions. See e.g. [12, 13] for some hints in this connection.

Given $\psi(x) = C\sqrt{(1-x^2)\cos(\alpha x)}$, $|x| \le 1$, we would like to know whether the Dirichlet boundary data (e.g. vanishing of a function for $|x| \ge 1$) are respected by $A_D\,\psi(x)$. To this end let us consider

$$\frac{1}{\pi}\int_{-x-1}^{-x+1} \frac{\psi(x)-\psi(x+z)}{z^2}dz = \frac{1}{\pi}\int_{-x-1}^{-x+1} \frac{\psi(x) - C\sqrt{1-(x+z)^2}(1-\gamma_2(x+z)^2 - \gamma_4(x+z)^4 - \ldots)}{z^2}dz, \tag{14}$$

where we have expanded $\sqrt{\cos\alpha(x+z)}$ into power series whose coefficients are denoted γ_{2n}, $(\gamma_0 = 1)$.

First few coefficients are given explicitly,

$$\gamma_2 = \frac{\alpha^2}{4}, \quad \gamma_4 = \frac{\alpha^4}{96}, \quad \gamma_6 = \frac{19\alpha^6}{5760}, \quad \gamma_8 = \frac{559\alpha^8}{645120}. \tag{15}$$

We integrate each term of the series separately. The integral corresponding to $\gamma_0 = 1$ can be rewritten as follows (see e.g. [23])

$$\frac{\psi(x)}{\pi}\int_{-x-1}^{-x+1} \frac{1-\sqrt{p+qz+rz^2}}{z^2}dz, \tag{16}$$

where $p = 1/\cos(\alpha x)$, $q = -2x/(1-x^2)\cos(\alpha x)$, $r = -1/(1-x^2)\cos(\alpha x)$. We evaluate the integral in the sense of its Cauchy principal value (see e.g. [23]), temporarily skipping the factor $\psi(x)/\pi$:

$$\int_{-x-1}^{-x+1} \frac{1-\sqrt{p+qz+rz^2}}{z^2}dz = -\frac{2}{(1-x^2)} + \frac{\pi}{\sqrt{(1-x^2)\cos(\alpha x)}}. \tag{17}$$

We note that the first term in the above (after restoring ψ/π) cancels its negative in the defining expression (6) for $A_D\psi(x)$.

Subsequent integrals can be evaluated analogously, but with $\psi(x)$ fully incorporated in the integrated expressions. We merely disregard (but keep in mind) an omnipresent coefficient C/π and denote $a = 1 - x^2, b = -2x, c = -1$. With this proviso other integrals follow:

$$\gamma_2 \int_{-x-1}^{-x+1} \frac{\sqrt{a+bz+cz^2}(x+z)^2}{z^2} = \gamma_2 \int_{-1}^{1} \frac{u^2\sqrt{1-u^2}}{(u-x)^2} du = \gamma_2 \frac{\pi(1-6x^2)}{2}, \tag{18}$$

$$\gamma_4 \int_{-x-1}^{-x+1} \frac{\sqrt{a+bz+cz^2}(x+z)^4}{z^2} = \gamma_4 \int_{-1}^{1} \frac{u^4\sqrt{1-u^2}}{(u-x)^2} du = \frac{\gamma_4}{8}\pi(1+12x^2-40x^4), \tag{19}$$

$$\gamma_6 \int_{-x-1}^{-x+1} \frac{\sqrt{a+bz+cz^2}(x+z)^6}{z^2} = \gamma_6 \int_{-1}^{1} \frac{u^6\sqrt{1-u^2}}{(u-x)^2} du = \frac{\gamma_6}{16}\pi(1+6x^2+40x^4-112x^6). \tag{20}$$

Accordingly, after reintroducing the factor C/π we arrive at the polynomial expansion of $A_D\psi$

$$A_D\psi(x) = C\sum_{n=0}^{\infty} \gamma_{2n} w_{2n}(x), \tag{21}$$

where coefficients γ_{2n} for $0 \le n \le 4$ have been explicitly identified before while $w_{2n}(x)$ are polynomials of degree $2n$, like e.g.

$$w_0(x) = 1, \quad w_2(x) = \frac{1-6x^2}{2}, \quad w_4(x) = \frac{1+12x^2-40x^4}{8}, \quad w_6(x) = \frac{1+6x^2+40x^4-112x^6}{16}. \tag{22}$$

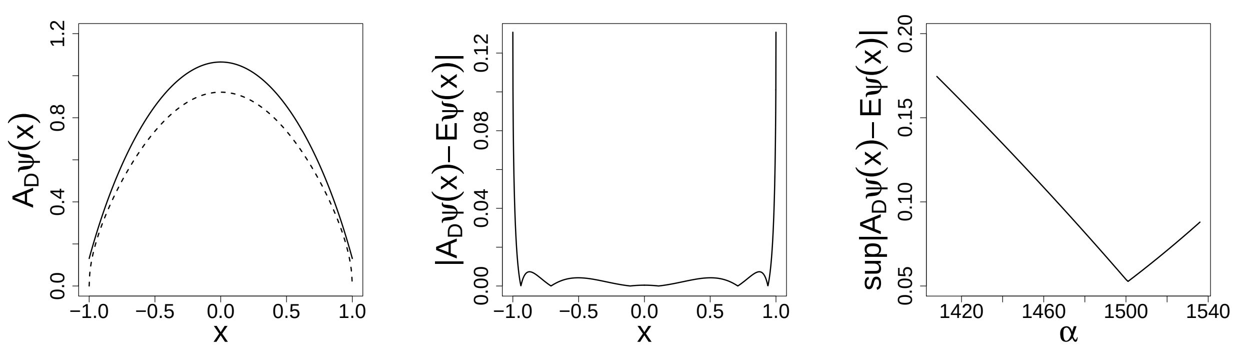

FIG. 2: Left panel: a comparison of $\psi(x) = C\sqrt{(1-x^2)cos(\alpha x)}$ (dotted line) and $A_D\psi$ (solid line). Middle panel: $|A_D\psi(x) - E\psi(x)|$ with $E = 1.156$. Right panel: supremum of $|A_D\,\psi - E\,\psi(x)|(\alpha)$ for $E = 1.156$. The α-axis is scaled in units $\pi/4096$.

The series expansion of $\sqrt{\cos(\alpha x)}$ converges very fast for $x \in [-1, 1]$. Accordingly, by accounting for only first few expansion terms we get quite good approximation of $A_D\,\psi$. In Fig. 2, we have depicted both $\psi(x) = C\sqrt{(1-x^2)cos(\alpha x)}$ (a dotted line) and the resultant $A_D\,\psi$ (solid line).

Since in Ref. [15] we have numerically identified an approximate ground state eigenvalue to be close to $E = 1.156$ (eventually correctable to $E = 1.1573$, [15]), let us employ the latter value instead of less accurate $E = 1.15318$ obtained in a rough reasoning presented before. Then, we can point-wise compare $\psi(x)$ against $A_D\psi(x)$ by depicting a curve $|A_D\psi(x) - E\psi(x)|, x \in D$, see e.g. Fig. 3.

The deviation of $A_D\psi(x)$ form $E\,\psi(x)$ appears to be rather small and effectively concentrates in the vicinity of the boundaries $x = \pm 1$. Since we have $\lim_{x\to\pm 1} A_D\psi(x) = 0.130753$, $\lim_{x\to\pm 1} E\psi(x) = 0$, there holds $|A_D\psi(x) - E\psi(x)| \leqslant 0.130753$, $x \in D$ which is the best point-wise estimate ever obtained in the literature on the (shape) subject, compare e.g. [11] (Lemma 1, formulas 8.9 and 8.10) and [8]. We emphasize that the behavior of $|A_D\psi(x) - E\psi|$ is fairly robust with respect to the specific choice of E. The dominant contribution comes to the upper bound comes form the behavior of $A_D\,\psi(x)$ at the boundaries od D.

The shape of the approximate ground state, as proposed in [8, 11], while away from the boundaries (i.e. around $x = 0$) is significantly different from our present finding and from the numerically-deduced behavior of eigenfunctions in finite but deep Cauchy wells, [15].

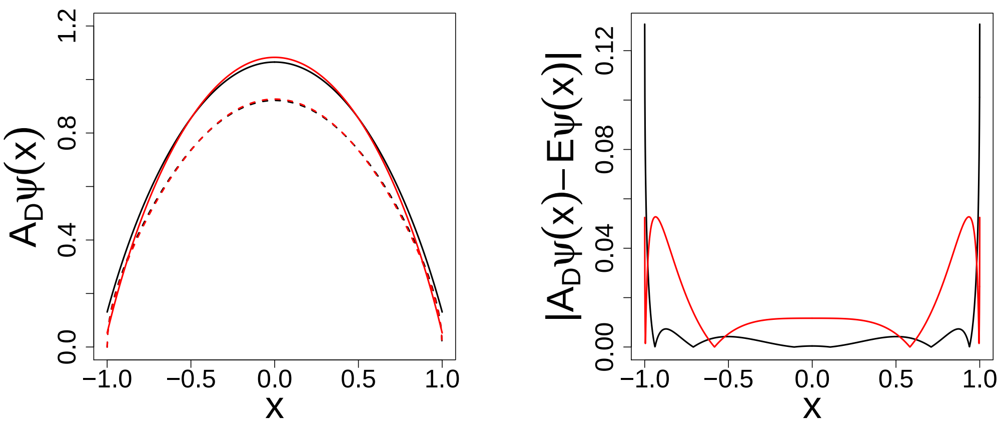

FIG. 3: Left panel: $\psi(x)$ is represented by dotted lines (black and red variants are practically indistinguishable), $A_D\psi(x)$ for $\alpha = 1443\pi/4096$ (solid black) and $\alpha = 1501\pi/4096$ (solid red). Right panel: $|A_D\psi(x) - E\psi(x)|$ for $E = 1.156$ and previous αs, respectively in black and red.

We note that in the definition (11) we can in principle vary α. Taking a supremum of $|A_D\,\psi(x) - E\psi(x)|(\alpha)$ over $x \in D$ as a criterion for how close $A_D\psi$ is to $E\psi$, we realize that the optimal α choice would be $\alpha = 1501\pi/4096$, see e.g. at the location of the minimum in Fig. 4.

To see better how the choice of α may affect the shape of $A_D\psi(x)$ and $|A_D\psi(x) - E\psi(x)|$, we display the behavior of these functions comparatively for $\alpha = 1443\pi/4096$ and $\alpha = 1501\pi/4096$. We note that for $\alpha = 1501\pi/4096$ we have $|A_D\psi(x) - E\psi(x)| < 0.06$ which is much better point-wise estimate than previously obtained 0.13 (for $\alpha = 1443\pi/4096$). The price paid is slightly worse fitting away form the ± 1 boundaries.

C. First excited state.

The ground state function, previously denoted $\psi(x)$, in fact should be labeled by $n = 1$, hence denoted $\psi_1(x)$. To avoid notational confusion, the first excited state ($n = 2$) will be denoted $\psi_2(x)$.

We introduce a trial analytic approximation of a "true" first excited state in the form:

$$\psi_2(x) = -C \sin(\beta x)\sqrt{(1 - x^2)\cos(\beta x)}, \tag{23}$$

where

$$\beta = \frac{1760}{4096}\pi = \frac{\pi}{2} - \frac{\pi}{16} - \frac{\pi}{128}, \tag{24}$$

and $C = 1.99693$ is a normalization constant. The minus sign is basically irrelevant, but is introduced for graphical comparison purposes. The parameter β fitting comes from our data for deep but finite Cauchy wells, [15]. In Fig. 6 we display various approximate formulas for the "true" excited eigenfunction, comparing our analytic guess with approximate curves proposed in [8] and [15].

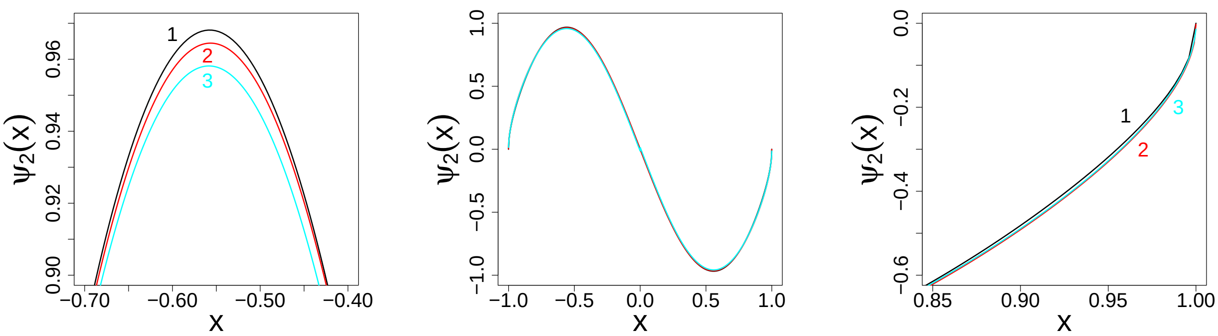

FIG. 4: Approximate expressions for the first excited state function. Numbers refer to: 1 - $\psi_2(x)$ (23), 2 - finite Cauchy well outcome, depth $V_0 = 5000$, [15], 3 - an approximation proposed in [8]. Left panel: magnified vicinity of a maxium. Right panel: magnified vicinity of the boundary $x = 1$.

D. Analysis of $A_D\,\psi_2(x)$.

Main arguments follow these of Section II.A and II.B. The first excited function $\psi_2(x)$ is odd, hence its power series expansion $\psi_2(x) = \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1}$ contains only odd-labeled coefficients c_n. Like in section IIA, we deduce the eigenvalue E_2

$$E_2 = \frac{4}{\pi}\left[1 - \frac{1}{c_1}\left(\frac{c_3}{1} + \frac{c_5}{3} + \frac{c_7}{5} + \ldots\right)\right]. \tag{25}$$

Since we know explicitly the numerical values of coefficients c_2, let us list few of them

$$\begin{array}{lllll} c_1 = -2.695662, & c_3 = 3.394555, & c_5 = -1.040718, & c_7 = 0.152499, & c_9 = 0.000755, \\ c_{11} = 0.010042, & c_{13} = 0.008479, & c_{15} = 0.007567, & c_{17} = 0.006774, & c_{19} = 0.006095, \end{array} \tag{26}$$

and next insert them directly to the expansion (26), while disregarding the remainder of the series. An approximate eigenvalue reads $E_2 = 2.72874$, to be compared (even though the result is very rough) with the deep finite Cauchy well prediction $E_2 = 2.7534$ of Ref. [15] (eventually correctable by 0.0013 to $E_2 = 2.7547$) and $E_2 = 2.7547$ of Ref. [8]. The series (26) converge slowly, therefore much larger number of coefficients c_n need to be accounted for, to make reliable the numerical value for E_2.

To deduce $A_D\psi_2(x)$, let us first analyze the integral

$$\int_{-x-1}^{-x+1} \frac{\psi_2(x) - \psi_2(x+z)}{z^2}dz \tag{27}$$

alone, see e.g. (6) for comparison. For clarity, in the power series expansion

$$\sin(\beta x)\sqrt{\cos(\beta x)} = \sum_{n=0}^{\infty} \gamma_{2n+1}x^{2n+1}, \quad x \in D, \tag{28}$$

we enlist few γ_{2n+1} in their explicit numerical form:

$$\gamma_1 = \beta, \quad \gamma_3 = -\frac{5\beta^3}{12}, \quad \gamma_5 = \frac{19\beta^5}{480}, \quad \gamma_7 = -\frac{31\beta^7}{8064}. \tag{29}$$

The Cauchy principal value can be evaluated for each expansion term of $\sin(\beta x)\sqrt{\cos(\beta x)}$ separately. Accordingly, for the first term we have

$$-C\gamma_1 \lim_{\varepsilon\to 0}\left(\int_{-x-1}^{-\varepsilon} + \int_{\varepsilon}^{-x+1}\right) \frac{x\sqrt{1-x^2} - (x+z)\sqrt{1-(x+z)^2}}{z^2}dz. \tag{30}$$

Since

$$\left(\int_{-x-1}^{-\varepsilon} + \int_{\varepsilon}^{-x+1}\right) \frac{x\sqrt{1-x^2}}{z^2} dz = x\sqrt{1-x^2}\left(-\frac{2}{1-x^2} + \frac{2}{\varepsilon}\right), \tag{31}$$

and

$$\left(\int_{-x-1}^{-\varepsilon} + \int_{\varepsilon}^{-x+1}\right) \frac{-(x+z)\sqrt{1-(x+z)^2}}{z^2} dz = -x\left(\frac{\sqrt{1-(x-\varepsilon)^2}}{\varepsilon} + \frac{\sqrt{1-(x+\varepsilon)^2}}{\varepsilon}\right) + 2\pi x + u(x,\varepsilon), \tag{32}$$

where (here unspecified) $u(x,\epsilon)$ approaches 0 if $\varepsilon \to 0$, for all $x \in D$.

The first term in (31), if multiplied by $-C\gamma_1/\pi$, cancels its negative in the expansion of $2\psi_2(x)/\pi(1-x^2)$. Because of

$$\lim_{\varepsilon \to 0}\left(\frac{2\sqrt{1-x^2} - \sqrt{1-(x-\varepsilon)^2} - \sqrt{1-(x+\varepsilon)^2}}{\varepsilon}\right) = 0, \tag{33}$$

the first expansion term has an ultimate form $-C\gamma_1 w_1(x)$, where $w_1(x) = 2x$.

In connection with (33) we point out that troublesome (divergent) $2/\varepsilon$ entries (related to the Cauchy principal value evaluation) are cancelled by their negatives coming from the principal value procedure of the form (5) while adopted to $-\int_{-x-1}^{-x+1} \frac{\psi_2(x+z)}{z^2} dz$. The remaining expansion terms of $\int_{-x-1}^{-x+1} \frac{\psi_2(x)}{z^2} dz$ are cancelled by their negatives that originate from $\psi_2(x)/(1-x^2)$.

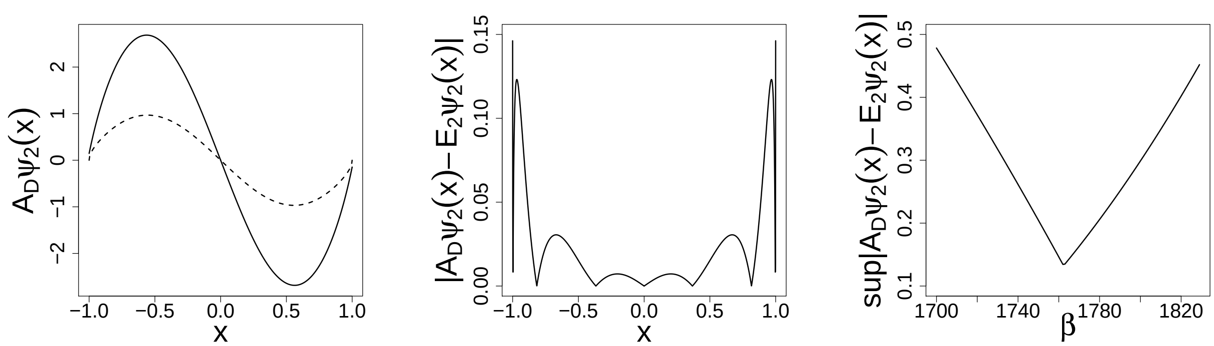

FIG. 5: Left panel: a comparison of the approximate function $\psi_2(x)$ (dotted line) with $A_D\psi_2(x)$ (solid line). Middle panel: we depict $|A_D\psi_2 - E_2\psi_2(x)|$, where $E = 2.75$. Right apnel: supremum of $|A_D\psi_2(x) - E_2\psi_2(x)|$, $E = 2.75$ as a function of β. The horizontal axis is scaled in units $\pi/4096$.

An ultimate form of $A_D\psi_2(x)$ is

$$A_D\psi_2(x) = \sum_{n=0}^{\infty} \gamma_{2n+1} w_{2n+1}(x), \quad x \in D, \tag{34}$$

where the coefficients γ_{2n+1}, $n = 0, 1, \ldots$ have been introduced before (while expanding $\sin(\beta x)\sqrt{\cos(\beta x)}$) and first few polynomials w_{2n+1} have the form:

$$w_1(x) = 2x, \quad w_3(x) = -x(1-4x^2), \quad w_5(x) = \frac{x(-1-8x^2+24x^4)}{4}, \quad w_7(x) = \frac{x(-1-4x^2-24x^4+64x^6)}{8}. \tag{35}$$

The convergence of (34) is much worse than that encountered in connection with the ground state function. Therefore a number of polynomials employed in the approximation of ψ_2 must be relatively large to make that approximation

reliable. In Fig. 7, we compare $A_D\psi_2(x)$ with $\psi_2(x)$, for an approximation restricted to first 15 series expansion terms only.

Like in case of the ground state function, we ask for an affinity of $A_D\psi_2(x)$ with $E_2\psi_2(x)$, where we adopt the value $E_2 = 2.75$. In Fig. 8 the affinity function $|A_D\psi_2(x) - E_2\psi_2(x)|$ is depicted and found to be bounded point-wise by 0.1462 which is much better estimate than any ever obtained, [8, 11].

In the definition of an approximate function $\psi_2(x)$ we have still some flexibility allowed with respect to the choice of the parameter β. In Fig. 9, $\sup|A_D\psi_2(x) - E_2\psi_2(x)|$ is depicted as a function of β in the vicinity of $\beta = 1760\pi/4096$. A minimum is achieved for $\beta = 1762\pi/4096$ and sets the upper bound 0.1344.

In principle, we can provide analytic approximations for (consecutive) higher excited state functions. However, our discussion of Section II should be considered merely as a warm-up, a preparatory step to address more serious goals.

Let us note that all ultimate formulas have involved the polynomial expansions. Interestingly, we could not associate them with any members of a hypergeometric family or other orthogonal polynomials in $D = (-1, 1)$). We shall follow this polynomial expansion strategy in the next section to get most accurate to date approximations of eigenvalues and eigenfunctions in the infinite Cauchy well problem.

We shall impose one more demand, deliberately absent in the existing literature on approximate Cauchy well eigenfunctions. We need that actually not only $\psi(x)$ is to vanish identically for $|x| \geq 1$, but $A_D\psi(x)$ as well, to become in all respects as close as possible to $E\,\psi(x)$, where E stands for an approximate eigenvalue.

III. POLYNOMIAL EXPANSIONS OF EIGENFUNCTIONS IN THE INFINITE CAUCHY WELL: PUSHING AHEAD APPROXIMATION FINESSE.

A. Ground state function addressed anew.

We wish to solve the eigenvalue problem $A_D\psi(x) = E\psi(x)$, with the Cauchy operator A_D in D, defined in Section II. This means that in the serch for approximate solutions, an approximation accuracy can be made arbitrarily fine, with the growth of the degree of the truncated polynomial expansion of the sought for eigenfunction. Functions in the domain of A_D are restricted by the exterior Dirichlet condition $\psi(x) = 0$, $|x| \geqslant 1$, but we impose the very same restriction upon the resultant $A_D\,\psi(x)$, given $\psi(x) \in D$.

We take Eq. (6) as a working definition of A_D and proceed with its integral part, here denoted

$$B_D\psi(x) = \frac{1}{\pi}\int\limits_{-x-1}^{-x+1} \frac{\psi(x) - \psi(x+z)}{z^2} dz. \tag{36}$$

Let us consider the action of B_D upon functions $\psi(x) = x^{2n}\sqrt{1-x^2}$. We get:

$$B_D\sqrt{1-x^2} = -\frac{2}{\pi}\frac{\sqrt{1-x^2}}{1-x^2} + 1, \tag{37}$$

$$B_D x^2\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^2\sqrt{1-x^2}}{1-x^2} - \frac{1-6x^2}{2}, \tag{38}$$

$$B_D x^4\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^4\sqrt{1-x^2}}{1-x^2} - \frac{1+12x^2-40x^4}{8}. \tag{39}$$

$$B_D x^6\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^6\sqrt{1-x^2}}{1-x^2} - \frac{1+6x^2+40x^4-112x^6}{16}. \tag{40}$$

Accordingly we have

$$B_D x^{2n}\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^{2n}\sqrt{1-x^2}}{1-x^2} + (c_{2n} + 3c_{2n-2}x^2 + 5c_{2n-4} + \ldots + (2n+1)c_0 x^{2n}), \tag{41}$$

where c_{2n} are expansion coefficients of the Taylor series for $\sqrt{1-x^2}$. Namely, we have

$$\sqrt{1-x^2} = \sum_{n=0}^{\infty} c_{2n}x^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{(1-2n)(n!)^2 4^n} x^{2n}, \tag{42}$$

which allows to rewrite $B_D x^{2n}\sqrt{1-x^2}$ as follows

$$B_D x^{2n}\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^{2n}\sqrt{1-x^2}}{1-x^2} + \sum_{k=0}^{n}\frac{(2k)!(2n+1-2k)}{(1-2k)(k!)^2 4^k}x^{2n-2k}. \tag{43}$$

Then ground state function is even, hence we can expect its power series expansion in the form:

$$\psi(x) = C\sqrt{1-x^2}\sum_{n=0}^{\infty}\alpha_{2n}x^{2n}, \qquad \alpha_0 = 1. \tag{44}$$

where our major task is to deduce the expansion coefficients α_{2n}.

Coming back to the definition (6), of A_D, we realize that $A_D\psi(x) = B_D\psi(x) + 2\psi(x)/\pi(1-x^2)$. The second term of this expression, upon employing the trial $\psi(x)$, as in Eq. (41) or (42), clearly cancels the first term of $B_D\psi(x)$, compare e.g. (36)-(42).

In view of this, the action of A_D upon the ground state candidate-function $\psi(x)$ of (44) greatly simplifies. Ultimately, the eigenvalue problem $A_D\psi(x) = E\psi(x)$ takes the form:

$$\sum_{n=0}^{\infty}\alpha_{2n}\sum_{k=0}^{n}\frac{(2k)!(2n+1-2k)}{(1-2k)(k!)^2 4^k}x^{2n-2k} = E\sum_{k=0}^{\infty}\frac{(2k)!}{(1-2k)(k!)^2 4^k}x^{2k}\sum_{n=0}^{\infty}\alpha_{2n}x^{2n}, \tag{45}$$

which can be re-ordered as follows

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\alpha_{2n}\frac{(2k)!(2n+1-2k)}{(1-2k)(k!)^2 4^k}x^{2n-2k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}E\alpha_{2n}\frac{(2k)!}{(1-2k)(k!)^2 4^k}x^{2k+2n}. \tag{46}$$

This system of equations (from which the coefficients α_{2n} are to follow for all n) looks hopelessly discouraging, if we are interested in a fully-fledged solution of the eigenvalue problem. However, things simplify if we look for approximate solutions, readily accessible upon a truncation of otherwise infinite series.

By definition we know that any solution $\psi(x)$ is defined in the domain $\bar{D} = [-1,1]$ and obeys the boundary condition $\psi(\pm 1) = 0$. We extend this restriction to $A_D\psi(x)$ and demand

$$\lim_{x\to\pm 1} A_D\psi(x) = 0. \tag{47}$$

In what follows we shall assume that $\alpha_0 = 1$. This choice is possible, in view of the presumed normalization (we have C involved). Each considered truncated series approximating $\psi(x)$ will be $l^2(D)$ normalized, this operation being encoded in a multiplicative constant C.

Example 1: Let us exemplify our procedure by truncating the series after the polynomial of degree 2. We have $A_D(1+\alpha_2 x^2)\sqrt{1-x^2} = 1 - \alpha_2\left(\frac{1}{2} - 3x^2\right)$ and the condition (47) implies $1 + \frac{5}{2}\alpha_2 = 0$. Accordingly we end up with an approximate eigenstate $\psi(x) = C(1 - 2/5x^2)\sqrt{1-x^2}$ where $C = \sqrt{875/996} \approx 0.937291$. The approximate eigenvalue reads $E = 1.2$.

Example 2: An analogous procedure for series terminating at the polynomial of the 4-th degree gives rise to $A_D(1+\alpha_2 x^2 + \alpha_4 x^4)\sqrt{1-x^2}$ and (47) implies $1 + \frac{5}{2}\alpha_2 + \frac{27}{8}\alpha_4 = 0$. Moreover, we have $1 - \frac{1}{2}\alpha_2 - \frac{1}{8}\alpha_4 = E$ and $3\alpha_2 - \frac{3}{2}\alpha_4 = E\left(-\frac{1}{2} + \alpha_2\right)$. The coefficients readily follow with values $\alpha_2 \approx -0.353189$ and $\alpha_4 \approx -0.0346746$. The approximate eigenvalue reads $E \approx 1.18093$. The normalized approximate eigenfunction takes the form $\psi(x) = C(1 - 0.353189x^2 - 0.0346746x^4)\sqrt{1-x^2}$, where $C = 0.931331$.

It is clear, that the procedure can be continued indefinitely by increasing the polynomial degree at the series truncation "point". An approximation accuracy grows with the polynomial degree. The polynomial degree growth increases the number of linear equations to solve.

Let $(a_{k,n})$ denote a matrix with elements

$$a_{k,n} = (2n+1-2k)c_k = \frac{(2k)!(2n+1-2k)}{(1-2k)(k!)^2 4^k}, \qquad n \geqslant k. \tag{48}$$

We recall that

$$c_k = \frac{(2k)!}{(1-2k)(k!)^2 4^k}. \tag{49}$$

If we consider an approximation of $\psi(x)$ by series terminating at the polynomial of degree $2n$, the eigenvalue problem we address takes the form of a linear system of equations with unknown E and α_{2n} (we recall our choice of $\alpha_0 = 1$):

$$\sum_{k=i}^{n} \alpha_{2k} a_{k-i,k} = E \sum_{k=0}^{i} \alpha_{2k} c_{i-k}, \qquad i = 0, 1, \ldots, n-1,$$
$$\sum_{m=0}^{n} \left(\alpha_{2m} \sum_{k=0}^{m} a_{k,m} \right) = 0. \tag{50}$$

The last identity in (50) comes from our demand (47), here adopted to $A_D w_{2n} \sqrt{1-x^2} = 0$ at $x = \pm 1$.

-	C	E	α_2	α_4	α_6	α_8	α_{10}	α_{12}	α_{14}	α_{16}
w_2	0.937291	1.200000	-0.400000	-	-	-	-	-	-	-
w_4	0.931331	1.180929	-0.353189	-0.03467461	-	-	-	-	-	-
w_6	0.927253	1.170127	-0.333863	-0.00891937	-0.0332900	-	-	-	-	-
w_8	0.925363	1.165443	-0.326159	-0.00332500	-0.0173088	-0.0221718	-	-	-	-
w_{10}	0.924339	1.162981	-0.322268	-0.00097523	-0.0134732	-0.0111668	-0.0163303	-	-	-
w_{12}	0.923728	1.161534	-0.320035	0.00025555	-0.0117497	-0.0084661	-0.0081667	-0.0126748	-	-
w_{14}	0.923337	1.160614	-0.318637	0.00098488	-0.0107978	-0.0072137	-0.0061348	-0.0063114	-0.0102120	-
w_{16}	0.923071	1.159993	-0.317704	0.00145367	-0.0102098	-0.0065016	-0.0051721	-0.0047139	-0.0050726	-0.0084590
w_{18}	0.922884	1.159555	-0.317051	0.00177313	-0.0098192	-0.0060507	-0.0046131	-0.0039456	-0.0037753	-0.0041958
w_{20}	0.922746	1.159234	-0.316576	0.00200068	-0.0095458	-0.0057448	-0.0042525	-0.0034927	-0.0031448	-0.0031160
w_{30}	0.922409	1.158447	-0.315422	0.00253637	-0.0089180	-0.0050710	-0.0035043	-0.0026342	-0.0021154	-0.0017921
w_{40}	0.922868	1.158159	-0.315006	0.00272257	-0.0087053	-0.0048519	-0.0032737	-0.0023882	-0.0018495	-0.0015006
w_{50}	0.922230	1.158022	-0.314810	0.00280842	-0.0086084	-0.0047537	-0.0031724	-0.0022828	-0.0017390	-0.0013839
w_{60}	0.922198	1.157948	-0.314703	0.00285494	-0.0085562	-0.0047014	-0.0031190	-0.0022279	-0.0016822	-0.0013250
w_{70}	0.922179	1.157902	-0.314638	0.00288292	-0.0085249	-0.0046702	-0.0030874	-0.0021957	-0.0016492	-0.0012910
w_{80}	0.922166	1.157872	-0.314595	0.00290106	-0.0085047	-0.0046501	-0.0030671	-0.0021751	-0.0016283	-0.0012696
w_{90}	0.922158	1.157852	-0.314566	0.00291348	-0.0084909	-0.0046364	-0.0030534	-0.0021612	-0.0016142	-0.0012552
w_{100}	0.922152	1.157837	-0.314545	0.00292235	-0.0084810	-0.0046267	-0.0030437	-0.0021514	-0.0016042	-0.0012451
w_{150}	0.922137	1.157802	-0.314496	0.00294331	-0.0084578	-0.0046039	-0.0030208	-0.0021285	-0.0015811	-0.0012218
w_{200}	0.922132	1.157789	-0.314478	0.00295063	-0.0084497	-0.0045959	-0.0030130	-0.0021206	-0.0015732	-0.0012139
w_{300}	0.922129	1.157781	-0.314466	0.00295585	-0.0084440	-0.0045903	-0.0030074	-0.0021151	-0.0015677	-0.0012083
w_{400}	0.922127	1.157778	-0.314461	0.00295767	-0.0084419	-0.0045884	-0.0030055	-0.0021132	-0.0015658	-0.0012064
w_{500}	0.922127	1.157776	-0.314459	0.00295851	-0.0084410	-0.0045875	-0.0030046	-0.0021123	-0.0015649	-0.0012056

TABLE I: Approximate solutions of the eigenvalue equation $A_D\psi(x) = E\psi(x)$. The approximating polynomial of degree $2n$ is indicated by $w_{2n}(x)$. We report first few values of coefficients α_{2k} for each $2n$-th case, together with an approximate eigenvalue E and the normalization coefficient C. For comparison we report the ground state eigenvalue reported in Ref. [11], E= 1.157773883697 (based on a diagonalization of the 900×900-matrix). Our ultimate result actually is $E = 1.1577764$.

The system (50) can be solved for various series truncation choices, up to the $2n = 500$ polynomial degree. All computations have been carried out by employing the routines of Wolfram Mathematica, which appear to be dedicated to solving even very large linear systems of equations.

Remark: One needs to be aware that (50), as a system of $n + 1$ equations, has more than one solution. We select an optimal approximation of the ground state function by selecting a solution with then least value of E. The same system of equations produces solutions that approximate higher (excited) even eigenfunctions. There appear also complex solutions which we discard as physically irrelevant.

Our findings are gathered in Table I, where we report explicit values for first few expansion coefficients α_{2n} of approximating polynomials, the approximate eigenvalue E and related normalization constant C. Symbols w_{2n} refer to approximating polynomials of degree $2n$. The computed eigenvalues definitely drop down with the growth of the polynomial degree $2n$, with a visible stabilization tendency. In our opinion, our result is much sharper than that

n	1	2	3	4	5	6	7	8	9	10
α_{2n}	-0.3144595	0.00295851	-0.0084410	-0.0045875	-0.0030046	-0.0021123	-0.0015649	-0.0012056	-0.0009571	-0.0007784
α_{20+2n}	-0.0006454	-0.0005439	-0.0004647	-0.0004016	-0.0003506	-0.0003088	-0.0002740	-0.0002449	-0.0002201	-0.0001990
α_{40+2n}	-0.0001808	-0.0001650	-0.0001512	-0.0001391	-0.0001284	-0.0001189	-0.0001104	-0.0001029	-0.0000960	-0.0000899
α_{60+2n}	-0.0000843	-0.0000793	-0.0000747	-0.0000705	-0.0000666	-0.0000631	-0.0000598	-0.0000568	-0.0000540	-0.0000515
α_{80+2n}	-0.0000491	-0.0000469	-0.0000448	-0.0000429	-0.0000411	-0.0000394	-0.0000378	-0.0000363	-0.0000349	-0.0000336
α_{100+2n}	-0.0000324	-0.0000312	-0.0000301	-0.0000291	-0.0000281	-0.0000272	-0.0000263	-0.0000255	-0.0000247	-0.0000239
α_{120+2n}	-0.0000232	-0.0000225	-0.0000219	-0.0000213	-0.0000207	-0.0000201	-0.0000196	-0.0000191	-0.0000186	-0.0000181
α_{140+2n}	-0.0000177	-0.0000172	-0.0000168	-0.0000164	-0.0000160	-0.0000157	-0.0000153	-0.0000150	-0.0000147	-0.0000143
α_{160+2n}	-0.0000140	-0.0000137	-0.0000135	-0.0000132	-0.0000129	-0.0000127	-0.0000125	-0.0000122	-0.0000120	-0.0000118
α_{180+2n}	-0.0000116	-0.0000114	-0.0000112	-0.0000110	-0.0000108	-0.0000106	-0.0000104	-0.0000103	-0.0000101	$-9.97 * 10^{-6}$
α_{200+2n}	$-9.81 * 10^{-6}$	$-9.67 * 10^{-6}$	$-9.53 * 10^{-6}$	$-9.39 * 10^{-6}$	$-9.26 * 10^{-6}$	$-9.13 * 10^{-6}$	$-9.00 * 10^{-6}$	$-8.88 * 10^{-6}$	$-8.76 * 10^{-6}$	$-8.65 * 10^{-6}$
α_{220+2n}	$-8.54 * 10^{-6}$	$-8.43 * 10^{-6}$	$-8.33 * 10^{-6}$	$-8.23 * 10^{-6}$	$-8.13 * 10^{-6}$	$-8.03 * 10^{-6}$	$-7.94 * 10^{-6}$	$-7.85 * 10^{-6}$	$-7.76 * 10^{-6}$	$-7.68 * 10^{-6}$
α_{240+2n}	$-7.59 * 10^{-6}$	$-7.51 * 10^{-6}$	$-7.43 * 10^{-6}$	$-7.36 * 10^{-6}$	$-7.28 * 10^{-6}$	$-7.21 * 10^{-6}$	$-7.14 * 10^{-6}$	$-7.08 * 10^{-6}$	$-7.01 * 10^{-6}$	$-6.95 * 10^{-6}$
α_{260+2n}	$-6.88 * 10^{-6}$	$-6.82 * 10^{-6}$	$-6.77 * 10^{-6}$	$-6.71 * 10^{-6}$	$-6.65 * 10^{-6}$	$-6.60 * 10^{-6}$	$-6.55 * 10^{-6}$	$-6.50 * 10^{-6}$	$-6.45 * 10^{-6}$	$-6.40 * 10^{-6}$
α_{280+2n}	$-6.35 * 10^{-6}$	$-6.31 * 10^{-6}$	$-6.27 * 10^{-6}$	$-6.22 * 10^{-6}$	$-6.18 * 10^{-6}$	$-6.14 * 10^{-6}$	$-6.10 * 10^{-6}$	$-6.07 * 10^{-6}$	$-6.03 * 10^{-6}$	$-6.00 * 10^{-6}$
α_{300+2n}	$-5.96 * 10^{-6}$	$-5.93 * 10^{-6}$	$-5.90 * 10^{-6}$	$-5.87 * 10^{-6}$	$-5.84 * 10^{-6}$	$-5.81 * 10^{-6}$	$-5.79 * 10^{-6}$	$-5.76 * 10^{-6}$	$-5.73 * 10^{-6}$	$-5.71 * 10^{-6}$
α_{320+2n}	$-5.69 * 10^{-6}$	$-5.67 * 10^{-6}$	$-5.65 * 10^{-6}$	$-5.63 * 10^{-6}$	$-5.61 * 10^{-6}$	$-5.59 * 10^{-6}$	$-5.57 * 10^{-6}$	$-5.56 * 10^{-6}$	$-5.54 * 10^{-6}$	$-5.53 * 10^{-6}$
α_{340+2n}	$-5.51 * 10^{-6}$	$-5.50 * 10^{-6}$	$-5.49 * 10^{-6}$	$-5.48 * 10^{-6}$	$-5.47 * 10^{-6}$	$-5.46 * 10^{-6}$	$-5.45 * 10^{-6}$	$-5.45 * 10^{-6}$	$-5.44 * 10^{-6}$	$-5.44 * 10^{-6}$
α_{360+2n}	$-5.43 * 10^{-6}$	$-5.43 * 10^{-6}$	$-5.43 * 10^{-6}$	$-5.43 * 10^{-6}$	$-5.43 * 10^{-6}$	$-5.43 * 10^{-6}$	$-5.43 * 10^{-6}$	$-5.44 * 10^{-6}$	$-5.44 * 10^{-6}$	$-5.45 * 10^{-6}$
α_{380+2n}	$-5.46 * 10^{-6}$	$-5.46 * 10^{-6}$	$-5.47 * 10^{-6}$	$-5.48 * 10^{-6}$	$-5.49 * 10^{-6}$	$-5.51 * 10^{-6}$	$-5.52 * 10^{-6}$	$-5.54 * 10^{-6}$	$-5.56 * 10^{-6}$	$-5.57 * 10^{-6}$
α_{400+2n}	$-5.59 * 10^{-6}$	$-5.62 * 10^{-6}$	$-5.64 * 10^{-6}$	$-5.66 * 10^{-6}$	$-5.69 * 10^{-6}$	$-5.72 * 10^{-6}$	$-5.75 * 10^{-6}$	$-5.78 * 10^{-6}$	$-5.82 * 10^{-6}$	$-5.86 * 10^{-6}$
α_{420+2n}	$-5.90 * 10^{-6}$	$-5.94 * 10^{-6}$	$-5.98 * 10^{-6}$	$-6.03 * 10^{-6}$	$-6.08 * 10^{-6}$	$-6.13 * 10^{-6}$	$-6.19 * 10^{-6}$	$-6.25 * 10^{-6}$	$-6.32 * 10^{-6}$	$-6.39 * 10^{-6}$
α_{440+2n}	$-6.46 * 10^{-6}$	$-6.54 * 10^{-6}$	$-6.62 * 10^{-6}$	$-6.71 * 10^{-6}$	$-6.81 * 10^{-6}$	$-6.91 * 10^{-6}$	$-7.03 * 10^{-6}$	$-7.15 * 10^{-6}$	$-7.28 * 10^{-6}$	$-7.42 * 10^{-6}$
α_{460+2n}	$-7.57 * 10^{-6}$	$-7.74 * 10^{-6}$	$-7.92 * 10^{-6}$	$-8.12 * 10^{-6}$	$-8.35 * 10^{-6}$	$-8.59 * 10^{-6}$	$-8.87 * 10^{-6}$	$-9.18 * 10^{-6}$	$-9.54 * 10^{-6}$	$-9.95 * 10^{-6}$
α_{480+2n}	-0.0000104	-0.0000110	-0.0000117	-0.0000125	-0.0000136	-0.0000150	-0.0000171	-0.0000204	-0.0000271	-0.0000540

TABLE II: For the approximate ground state function, the corresponding polynomial w_{500} is displayed in detail, in terms of its expansion coefficients α_{2n}. Note a numbering convention: in the first row we have displayed consecutively $\alpha_2, \alpha_4 ...$ up to α_{20}.

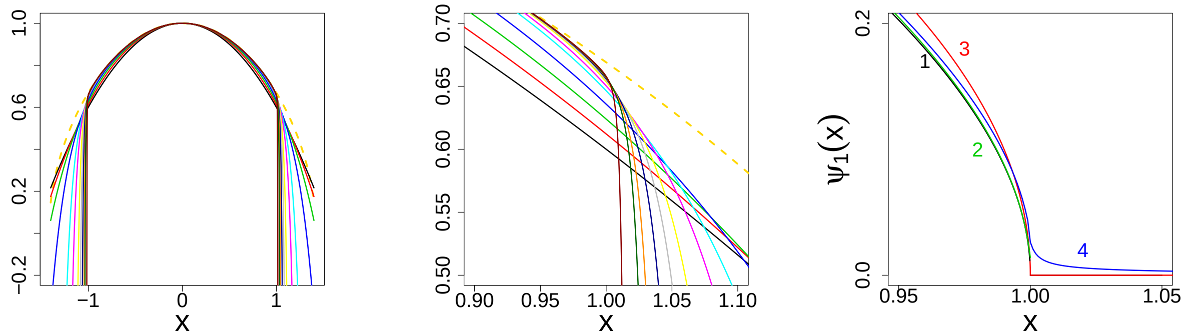

FIG. 6: Left panel: a comparative display of polynomials $w_{2n}(x)$ of degrees $2, 4, 6, 10, 20, 30, 50, 70, 100, 150, 200, 500$ and the curve $\sqrt{\cos{(1443\pi x/4096)}}$ (gold) which has been a building block in the formula (11). Middle panel provides an enlargement in the vicinity of the right boundary. Right panel depicts various approximations of the ground state function at the right boundary $x = 1$: 1 - curve $Cw_{500}(x)\sqrt{1-x^2}$, 2 - curve of [8], 3 - $\psi_1(x) \sim (1-|x|)^{1/2}$ of [2], 4 - $V_0 = 500$ finite well ground state of [15].

reported in Ref. [11], see also [8]. Numerical values of the coefficients α_{2n} grow as well with $2n$ growth, with a clear stabilization tendency.

In Table II we make explicit the functional form of the polynomial w_{500}. All coefficients α_{2n} are reproduced as well. At the moment that provides the best available approximation of the ground state function in the infinite Cauchy well problem.

Since we are interested in fine details of the eigenfunction shape, it is instructive to display the behavior of the major eigenfunction building blocks, i.e. the polynomials $w_{2n}(x)$, in the vicinity of the boundaries of D. In Fig. 6, we admit x from the exterior of $\bar{D}$, i.e. $|x| > 1$. We compare the near-the-boundaries behavior of polynomials of degrees $2, 4, 6, 10, 20, 30, 50, 70, 100, 150, 200, 500$, with a function $\sqrt{\cos{(1443\pi x/4096)}}$ (colored gold) appearing in the definition of the trial ground state function (11) (c.f. Section II). That clearly explains deficiencies of the cosine factor in the adopted definition and obvious virtues of the present polynomial expansion method. We note that the polynomial degree growth, is accompanied by a steeper decent of the representative curves at the boundaries.

At this point it is also instructive to have a comparison of the boundary behavior of the approximate ground state function proposed in the literature so far. We note that $\psi(x)$ of Ref. [8], at the boundaries, is fapp (for all practical purposes) identical with our $\psi(x) \sim Cw_{500}(x)\sqrt{1-x^2}$.

As before, we can analyze a deviation of $A_D\psi(x)$ from $E\,\psi(x)$ in terms of $|A_D\psi(x) - E\psi(x)|, \quad x \in \bar{D}$. Results are depicted in Fig. 7. Note that for the polynomial of degree $2n = 500$ we get an upper bound $|A_D\psi(x) - E\psi(x)| < 0.01$. We note that with the growth of the polynomial degree, the near-the-boundary maximum of $|A_D\psi(x) - E\psi(x)|$ decreases.

B. Other even eigenfunctions $\psi_{2k+1}(x), \quad k > 1$.

The system (50) of equations has been dedicated to obtain even eigenfunctions. As mentioned before it has infinitely many solutions, both real and complex-valued. Each real solution is interpreted as an approximation of a certain eigenfunction. Since for each resolved polynomial of degree $2n$, we can jointly compute approximate eigenfunctions and the corresponding eigenvalues, there appears a natural ordering with respect to increasing E-values which we enumerate by consecutive odd numbers $2k$, with $k = 1$ corresponding to the ground state. That allows for a systematic selection of higher rank even eigenfunctions. We keep intact the notation $w_{2n}(x)$ for an approximating polynomial of degree $2n$, although one should keep in mind that for each consecutive E_{2k+1}, we deal we the corresponding $2k+1$-th polynomial (and appropriate $2k+1$-th set of expansion coefficients). The previous Table II data refer to the ground state solution $\psi_1(x)$ only. The coefficients tables for other polynomials area available upon request.

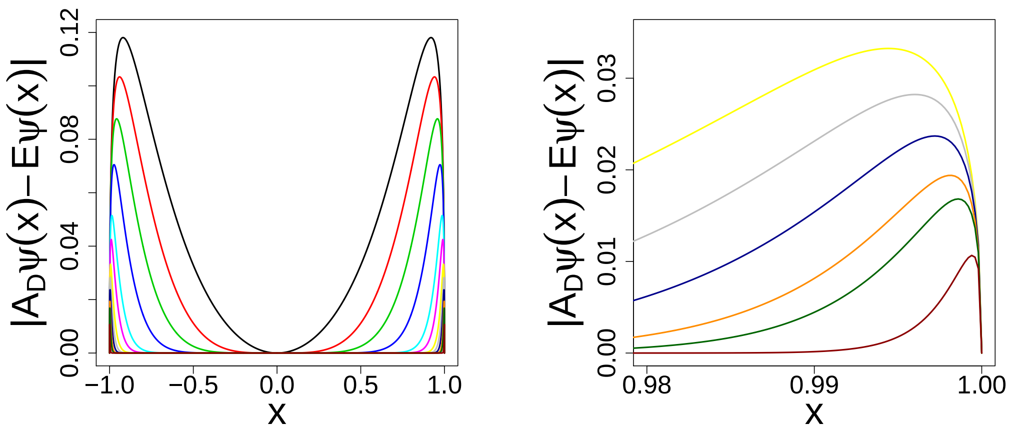

FIG. 7: Left panel: $|A_D\psi(x) - E\psi(x)|$ where $\psi = C\sqrt{1-x^2}w_{2n}(x)$, with $2n = 2, 4, 6, 10, 20, 30, 50, 70, 100, 150, 200, 500$. Right panel: polynomial degrees $2n = 50, 70, 100, 150, 200, 500$, $\psi(x)$ in the vicinity of the right boundary $x = 1$.

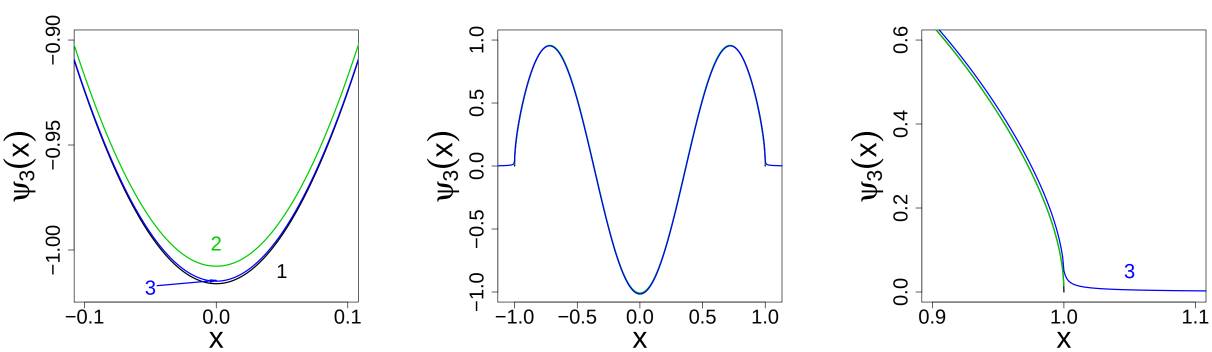

FIG. 8: An approximation of the third eigenfunction $\psi_3(x)$. Numbers refer to: 1 - $\psi_3 \sim Cw_{500}(x)\sqrt{1-x^2}$, 2 - $\psi_3(x)$ according to [8], 3 - finite $V_0 = 500$ Cauchy well ground state, [15]. Left panel: enlargement of the vicinity of the minimum. Right panel - enlargement of the vicinity of $x = 1$.

C. Odd eigenfunctions, $\psi_{2k}(x)$, $\quad k \geq 1$.

In the notation of Section III.A, the odd eigenfunctions case i can be handled by invoking:

$$B_D x\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x\sqrt{1-x^2}}{1-x^2} + 2x, \tag{51}$$

$$B_D x^3\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^3\sqrt{1-x^2}}{1-x^2} + 2x\left(-\frac{1}{2} + 2x^2\right), \tag{52}$$

$$B_D x^5\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^5\sqrt{1-x^2}}{1-x^2} + 2x\left(-\frac{1}{8} - 2\cdot\frac{1}{2}x^2 + 3x^4\right). \tag{53}$$

$$B_D x^7\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^7\sqrt{1-x^2}}{1-x^2} + 2x\left(-\frac{1}{16} - 2\cdot\frac{1}{8}x^2 - 3\cdot\frac{1}{2}x^4 + 4x^6\right). \tag{54}$$

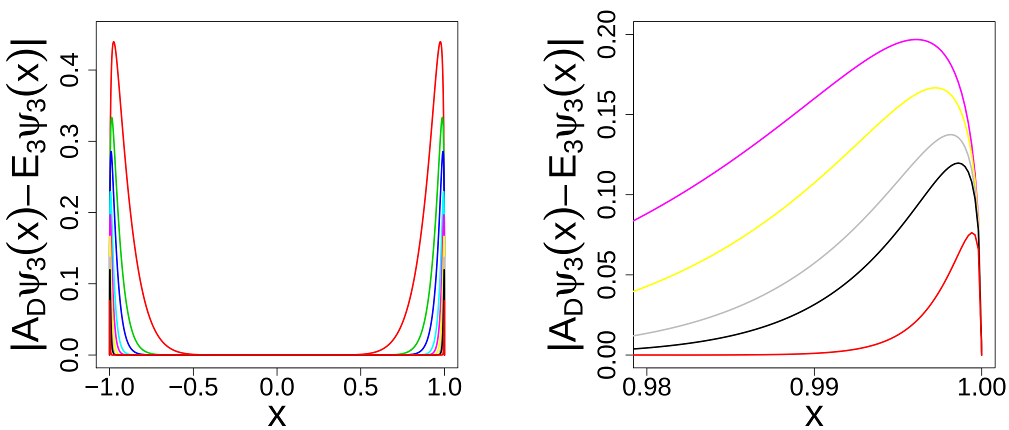

FIG. 9: $|A_D\psi_3(x) - E_3\psi_3(x)|$ where ψ_3 is a product of $C\sqrt{1-x^2}$ and a polynomial of degree $2n$, we depict $2n = 10, 20, 30, 50, 70, 100, 150, 200, 500$. Right panel refers to $2n = 70, 100, 150, 200, 500$. Note that for $2n = 500$, we have $|A_D\psi_3(x) - E_3\psi_3(x)| < 0.07$.

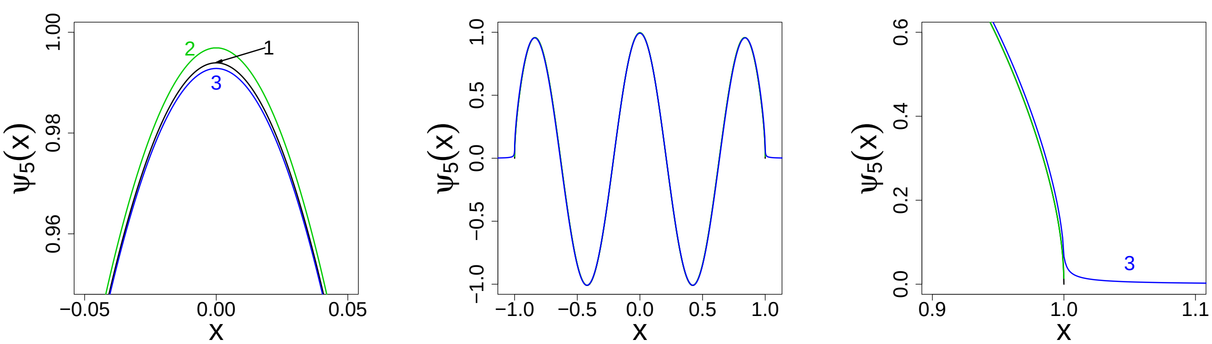

FIG. 10: For an approximate eigenfunction $\psi_5(x)$ we display: 1 - $2n = 500$, 2 - $\psi_5(x)$ of Ref. [8], 3 - the fifth finite $V_0 = 500$ Cauchy well eigenfucntion (computed, but not reproduced in [15]). Left panel - minimum vicinity enlargement. Right panel - $x = 1$ vicinity enlargement.

i.e.

$$B_D x^{2n+1}\sqrt{1-x^2} = -\frac{2}{\pi}\frac{x^{2n+1}\sqrt{1-x^2}}{1-x^2} + 2x(c_{2n} + 2c_{2n-2}x^2 + 3c_{2n-4} + \ldots + (n+1)c_0 x^{2n}), \tag{55}$$

where c_{2n} are Taylor series coefficients for $\sqrt{1-x^2}$. We are interested in odd eigenfunctions and seek them in the form:

$$\psi(x) = C\sqrt{1-x^2}\sum_{n=0}^{\infty}\beta_{2n+1}x^{2n+1}, \qquad \beta_1 = 1. \tag{56}$$

As in the case of even functions, we look for solutions of the eigenvalue problem $A_D\psi(x) = E\,\psi(x)$, so arriving at

$$\sum_{n=0}^{\infty}\beta_{2n+1}\sum_{k=0}^{n}\frac{(2k)!(2n+2-2k)}{(1-2k)(k!)^2 4^k}x^{2n-2k+1} = E\sum_{k=0}^{\infty}\frac{(2k)!}{(1-2k)(k!)^2 4^k}x^{2k}\sum_{n=0}^{\infty}\beta_{2n+1}x^{2n+1}, \tag{57}$$

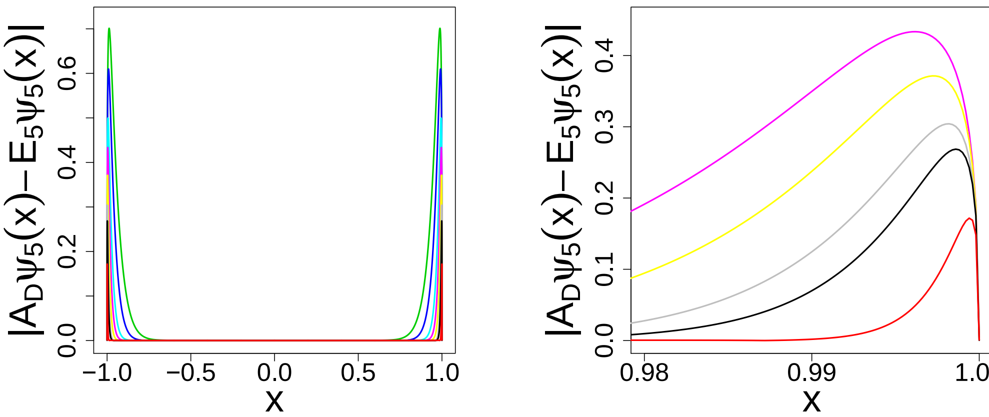

FIG. 11: $|A_D\psi_5(x) - E_5\psi_5(x)|$ where $\psi_5(x)$ is inferred for $2n = 20, 30, 50, 70, 100, 150, 200, 500$. Right panel - $2n = 70, 100, 150, 200, 500$.

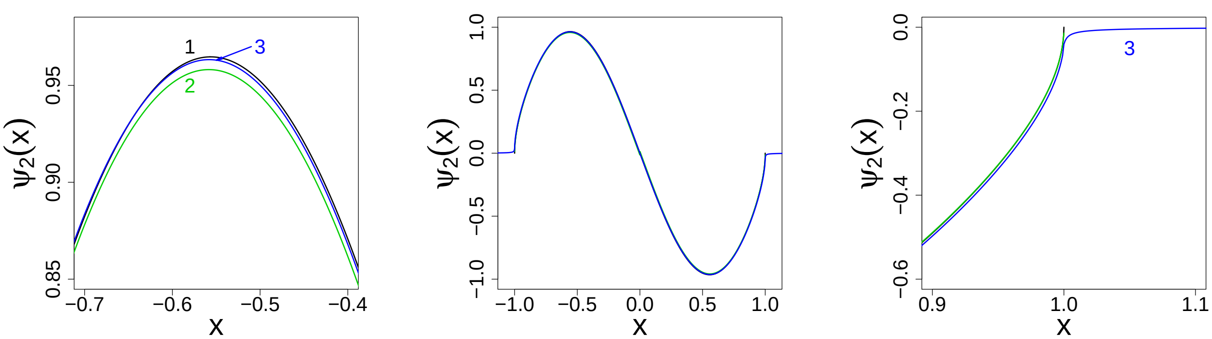

FIG. 12: $\psi_2(x)$, numbers refer to: 1 - polynomial of degree 501, 2 - according to [8], 3 - finite Cauchy well $V_0 = 500$, [15]. Left panel - enlargement of the minimum. Right panel - enlargement of the $x = 1$ boundary.

and next at

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\beta_{2n+1}\frac{(2k)!(2n+2-2k)}{(1-2k)(k!)^2 4^k}x^{2n-2k+1} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}E\,\beta_{2n+1}\frac{(2k)!}{(1-2k)(k!)^2 4^k}x^{2k+2n+1}. \tag{58}$$

Additionally, we impose our boundary condition

$$\lim_{x\to\pm 1} A_D\psi(x) = 0. \tag{59}$$

The procedure adopted to find polynomial approximations of eigenfunctions and eigenvalues in the even case, can be extended to the odd case as well. Let

$$b_{k,n} = (2n+2-2k)c_k = \frac{(2k)!(2n+2-2k)}{(1-2k)(k!)^2 4^k}, \qquad n \geqslant k, \tag{60}$$

where

$$c_k = \frac{(2k)!}{(1-2k)(k!)^2 4^k}. \tag{61}$$

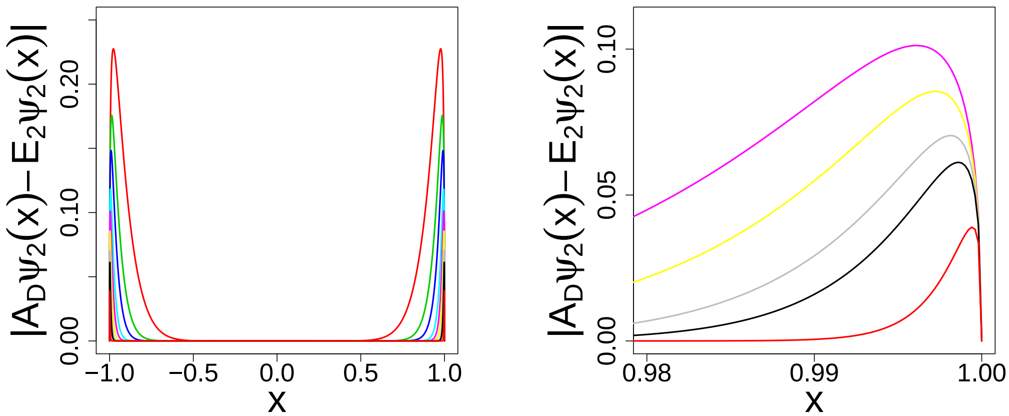

FIG. 13: $|A_D\psi_2(x) - E_2\psi_2(x)|$, for polynomial degrees $2n+1 = 11, 21, 31, 51, 71, 101, 151, 201, 501$. Right panel - $2n+1 = 71, 101, 151, 201, 501$.

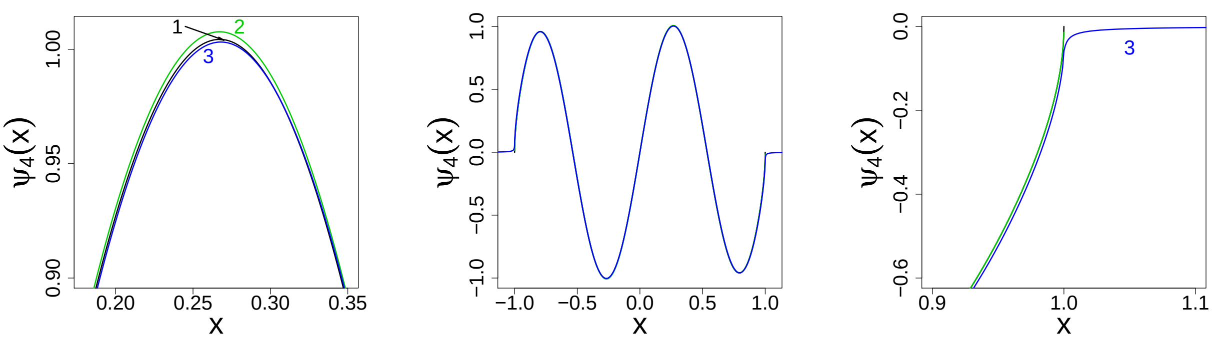

FIG. 14: $\psi_4(x)$, numbers refer to: 1 - polynomial of degree 501, 2 - $\psi_4(x)$ according to [8], 3 - finite Cauchy well $V_0 = 500$ outcome, [15]. Left panel - minimum enlargememnt. Right panel - right boundary enlargement.

The linear system of equations from which all β_{2n+1} and E are to be inferred has the form (we set $\beta_1 = 1$)

$$\sum_{k=i}^{n} \beta_{2k+1} b_{k-i,k} = E \sum_{k=0}^{i} \beta_{2k+1} c_{i-k}, \qquad i = 0, 1, \ldots, n-1,$$
$$\sum_{m=0}^{n} \left(\beta_{2m+1} \sum_{k=0}^{m} b_{k,m} \right) = 0. \tag{62}$$

Like in case of (50), the present system of equations has infinitely many solutions, real and complex-valued for each fixed n. As before, in the set of real solutions, an ordering relation is set by referring to an increasing order of computed eigenvalues $E_{2k} \quad k \geq 1$.

Skipping unnecessary repetitions of previous arguments, we display our findings concerning $\psi_2(x)$ and $\psi_4(x)$, together with estimates for $|A_D\psi(x) - E\,\psi(x)|$. A comparison is made with previously reported results on the shape of corresponding approximate eigenfunctions. It appears that our method provides most accurate to date data for both approximate eigenfunctions and eigenvalues and provides the sharpest to date point-wise estimates for $|A_D\psi(x) - E\psi(x)|$.

For completeness, in Table III we report first five computed eigenvalues ordered against the approximating polyno-

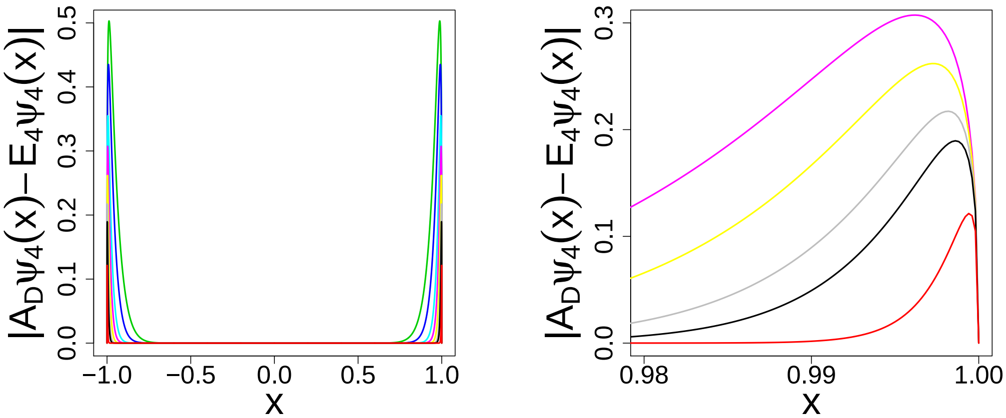

FIG. 15: $|A_D\psi_4(x) - E_4\psi_4(x)|$ where polynomial degrees are odd, $2n + 1 = 21, 31, 51, 71, 101, 151, 201, 501$. Right panel: $2n + 1 = 71, 101, 151, 201, 501$.

n	E_1	E_2	E_3	E_4	E_5
7	1.160614	2.768252	4.351150	5.946117	7.337136
8	1.159993	2.765561	4.344362	5.934918	7.584192
9	1.159555	2.763594	4.339381	5.928041	7.512343
10	1.159234	2.762114	4.335613	5.922546	7.509991
15	1.158447	2.758299	4.325845	5.907535	7.485347
20	1.158159	2.756826	4.322066	5.901342	7.475242
25	1.158022	2.756110	4.320233	5.898233	7.470144
30	1.157948	2.755709	4.319211	5.896463	7.467238
35	1.157902	2.755463	4.318584	5.895363	7.465432
40	1.157872	2.755301	4.318173	5.894634	7.464235
45	1.157852	2.755188	4.317889	5.894127	7.463403
50	1.157837	2.755107	4.317685	5.893759	7.462802
75	1.157802	2.754913	4.317196	5.892875	7.461356
100	1.157789	2.754844	4.317024	5.892559	7.460842
150	1.157781	2.754795	4.316900	5.892331	7.460473
200	1.157778	2.754777	4.316857	5.892251	7.460343
250	1.157776	2.754769	4.316837	5.892214	7.460282
K	1.157773	2.754754	4.316801	5.892147	7.460175

TABLE III: We display the computed eigenvalues E_k, $k \geq 1$ under an assumption that polynomials of degree $2n$ were employed in the definition of even eigenfunctions and $2n + 1$ for odd eigenfunctions. The capital K in the last line indicates data collected from Ref. [11].

mial degree and compare them with those obtained by other arguments in Refs. [8, 11]. It is clear that by increasing the polynomial degree n we can achieve still higher finesse level of a computational accuracy with which both eigenfunctions and eigenvalues can be retrieved. It is seen that with the growth n, there are definite stabilization symptoms in the numerical outcomes for the eigenvalues. We would like to point out that to increase the reproduction accuracy of higher "true" eigenfunctions, one should pass to higher than $n = 500$ polynomial degrees. The same pertains to the 0.01 upper bound for $A_D\psi(x) - E\psi(x)|$ if $\psi(x)$ is an approximation of the ground state. To push that bound closer to 0, higher polynomiola degrees are necessary.

IV. OUTLOOK

In the present paper we were largely motivated by: (i) on the one hand -successes in the approximate evaluation of eigenfunctions and eigenvalues for the infinite Cauchy well problem [8, 11, 15], (ii) various drawbacks in the physics-motivated procedures to solve that eigenvalue problem, summarized in [1, 15]. It has been often mentioned that the "true" eigenfunctions show a striking similarity to trigonometric sine or cosine functions (identified as eigenfunctions of the standard Laplacian in the interval) when away from the boundaries of D, while their fall-off towards zero at the boundaries should be similar to $\sqrt{1-x^2}$. Our trial function considerations of Section II proved that the trigonometric connection is somewhat deceiving, since the square root of a trigonometric function has been involved. In Section III we have resolved the away-from-the-boundary behavior by means of truncated polynomial expansions, that bear no obvious similarity, neither to trigonometric functions nor to their square roots. Things became more complicated and subtle, since the polynomial shapes actually dictate minute details of the eigenfunctions fall-off to 0 at the boundaries. In this connection, we point out the peculiar fall-off of approximating polynomials around ± 1, as depicted in Fig. 6.

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